

MAT 1341 Final Exam

Summer 2007

July 16

Instructor: Charles Starling

Family Name: _____

First Name: _____

Student Number: _____

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **Do not use your own scrap paper! Use the last page or the backs of pages for rough work.**
3. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.
4. Questions 1 through 10 are multiple choice. They are worth 2 points each and no part marks will be given. Please record your answers in the space provided above.
5. Questions 11 through 15 require a complete solution, and are worth 6 points each, so spend your time accordingly. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.** You must answer these questions in the space provided. Use the backs of pages if necessary.
6. **Question 16 is a bonus question**, worth 3 marks. Only attempt this if you have completed the exam and checked your work.
7. Again, where it is possible to check your work do so.
8. Good luck! Bonne chance!

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1. Let z_1 be a complex number, $a > 0$ be real and

$$S = \{z \in \mathbb{C} \mid |z - z_1| = a\}.$$

Then,

- A. S is a circle with centre at z_1 and radius a .
- B. S is a straight line passing through z_1 .
- C. S consists of all the points whose distance from z_1 is smaller than a .
- D. S consists of all the points whose distance from z_1 is greater than a .
- E. S is a circle centered at the origin with radius $|z_1|$.
- F. S is a pair of parallel lines.

2. Which two of the following are subspaces of $\mathbb{F}(\mathbb{R}) = \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$?

$$S = \{f \in \mathbb{F}(\mathbb{R}) \mid f(x) = f(2x) \text{ for all } x \in \mathbb{R}\}$$

$$T = \{f \in \mathbb{F}(\mathbb{R}) \mid f(0) > 0\}$$

$$U = \{f \in \mathbb{F}(\mathbb{R}) \mid f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$$

$$V = \{f \in \mathbb{F}(\mathbb{R}) \mid f(0) = 1\}$$

- A. V and T .
- B. T and U .
- C. S and T .
- D. V and S .
- E. S and U .
- F. V and U .

3. For a non-homogeneous system of 5 equations in 14 unknowns, answer the following three questions:

- Can the system be inconsistent?
- Can the system have a unique solution?
- Can the system have infinitely many solutions?

A. No, Yes, Yes.

B. Yes, Yes, No.

C. Yes, No, Yes.

D. No, No, Yes.

E. Yes, Yes, Yes.

F. No, Yes, No.

4. The dimension of the subspace of \mathbb{R}^5 spanned by the vectors $(1, 0, 3, 1, 1)$, $(1, -1, 7, -1, 0)$, $(2, 1, 2, 4, 3)$ and $(5, 1, 11, 7, 6)$ is:

A. 5

B. 4

C. 3

D. 2

E. 1

F. 0

5. The inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

A. $\begin{bmatrix} 1 & 0 & \frac{1}{b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $b \neq 0$. B. $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

C. $\begin{bmatrix} 1 & 0 & -\frac{1}{b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ if $b \neq 0$. D. $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

E. $\begin{bmatrix} b & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -b \end{bmatrix}$. F. $\begin{bmatrix} b & 0 & -1 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$.

6. Let \mathbb{M}_{22} denote the vector space of real 2×2 matrices and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The dimension of $C = \{A \in \mathbb{M}_{22} \mid JA = AJ\}$ is:

- A. 0
- B. 1
- C. 2
- D. 3
- E. 4
- F. 5

7. Let A be a 6×12 matrix such that $Ax = 0$ has 8 parameters in its general solution.

- What is the rank of A ?
- Is $Ax = b$ consistent for all $b \in \mathbb{R}^6$?

A. 2, Yes

B. 4, Yes

C. 6, Yes

D. 2, No

E. 4, No

F. 6, No

8. Let A and B denote matrices and x a column vector. If all products below are defined, which of the following statements are true?

(1) If either A or B is a multiple of the identity then $AB = BA$.

(2) If $AB = 0$, then either $A = 0$ or $B = 0$.

(3) If A is $m \times n$ and $m < n$ the system $Ax = 0$ has infinitely many solutions.

A. (1) and (2).

B. (1) and (3).

C. (2) and (3).

D. (1) only.

E. (2) only.

F. (3) only.

9. Compute the determinant $\begin{vmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{vmatrix}$

- A. -10
- B. 18
- C. 30
- D. 15
- E. -18
- F. -36

10. The matrix $A = \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix}$ is diagonalizable. Which of the following could be $P^{-1}AP$ for some invertible matrix P ?

A. $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ F. $\begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$

11. a) Suppose that A and B are square matrices and that AB is invertible. Prove that A and B are invertible.
- b) Suppose that A is an invertible matrix and that $AC = CA$ for some matrix C . Show that $A^{-1}C = CA^{-1}$.
- c) Suppose that B is a 2×2 matrix such that $AB = BA$ for every 2×2 matrix A . Show that $B = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some $a \in \mathbb{R}$.
- d) A matrix M is said to be *symmetric* if $M = M^t$. Show that if A , B and AB are symmetric matrices, then $AB = BA$.

12. Let $W = \text{span}\{(1, 0, 0, 1), (0, 1, 1, 0), (1, 0, 0, 0), (0, 1, 1, 1)\} \subset \mathbb{R}^4$.

- a) Find a basis of W which is a subset of the given spanning set.
- b) Find an orthogonal basis of W .
- c) Find the best approximation to $(0, 1, -1, 1)$ in W .
- d) Extend (if necessary) your basis of W in (b) to a basis of \mathbb{R}^4 .

13. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

a) Find the characteristic polynomial $c_A(\lambda) = \det(A - \lambda I)$ of A , and deduce that the eigenvalues of A are 1 and 4.

b) Find a basis of $E_1 = \{x \in \mathbb{R}^3 \mid Ax = 1x\}$.

c) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 4.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D . Explain why your choice of P is invertible.

14. Suppose $a, c \in \mathbb{R}$ and consider the following system in the unknowns x, y and z :

$$\begin{aligned}x &+ az = 1 \\2x + y + 2az &= 3 \\3x + y + 4az &= c\end{aligned}$$

a) If $[A \mid b]$ is the augmented matrix of the system above, find $\text{rank}(A)$ and $\text{rank}([A \mid b])$ for all values of a and c .

b) Using part (a), find all values of a and c so that this system has:

- a unique solution,
- infinitely many solutions, and
- no solutions.

c) In the second case in b) above (infinitely many solutions), give a geometric description of the set of solutions.

15. a) Let A be an $n \times n$ matrix. Give 4 statements equivalent to “ $\text{rank}(A) = n$ ”, in terms of:

i) the columns of A .

ii) the determinant of A .

iii) the homogeneous system $Ax = 0$.

iv) the reduced row-echelon (RRE) form of A .

15. b) State whether the following are true or false. You must justify your answer.

i) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

ii) The dimension of $\{v \in \mathbb{R}^5 \mid v \cdot (1, 2, 3, 4, 5) = 0\}$ is 4.

iii) If $\{v_1, v_2, v_3\}$ is linearly independent, then so is $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$.

iv) If A is an 11×11 matrix and $A^t = -A$, then $\det A = 0$.

16. (**BONUS**) Recall that if $A \sim A'$, then

$$\bullet \det(A') = (-1)^j k_1 k_2 \cdots k_m \det(A)$$

where j and k_1, k_2, \dots, k_m depend on the row operations that take you from A to A' .

Using this fact, prove that $\det(AB) = \det(A) \det(B)$ for two $n \times n$ matrices A and B by following these steps:

a) Prove that $\det(AB) = \det(A) \det(B)$ in the case where either A or B is not invertible.

b) In the case that A and B are both invertible, show that the augmented matrix $[A \mid AB]$ is row equivalent to $[I_n \mid B]$ (**hint**: use the fact that $\begin{bmatrix} Be_i \\ -e_i \end{bmatrix}$ is in $\ker [A \mid AB]$ for each standard basis vector e_i).

c) Use part b) and \bullet above to prove that $\det(AB) = \det(A) \det(B)$.

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